## **13. Green functions method for the Laplace and the Poisson equations**

Consider the Green formulas of the multidimensional integration theory. This is used for determining the Green functions of the Dirichlet and Neumann problems for the Laplace and the Poisson equations. We obtain the representation of the solutions of these boundary problems, using the Green functions methods.

### **13.1. Gauss – Ostrogradsky formula**

Our problem is to solve the differential equations. Solving of differential equations use the integration, because this is the inverse operation to the differentiation. If we solve the ordinary differential equation, we use the simple integration. This is true for the parabolic and the hyperbolic equations with unique spatial variable, because the time and the spatial variable are independent, and we can calculate the time integral and the spatial integral separately. However, for the elliptic equation this is not true. It is necessary to use the double integrals and the triple integrals. Therefore, it is necessary to have some additional information about multidimensional integrals for the analysis of the Laplace equation and its extension.

We know the relation between the differentiation and the integration. This is the ***Newton – Leibniz formula***

  (13.1)

for all differentiable functions *f.* This formula has a multidimensional analogue. Let Ω be the three-dimensional set with boundary *S.* For all smooth enough functions *R = R*(*x*,*y*,*z*), one knows the ***Gauss – Ostrogradsky formula***

  (13.2)

where *γ* is the angle between the direction of the axe *z* and the exterior normal to the surface *S.*

Compare the formulas (13.1) and (13.2). We have the integrals with respect to the considered set in the left hand-side for both cases. There are the interval [*a*,*b*] for the equality (13.1) and the set Ω for the equality (13.2). Besides, we have the first derivatives of the considered functions under these integral. Then there is the integral with respect to the boundary *S* of the given set for the second case. The set boundary for the first case consists of two points *a* and *b*. The finite analogue of the integration operation is addition. Therefore, we have the sum of the boundary value –*f*(*a*) and *f*(*b*) of the considered function. However, the value under the boundary integral of the equality (13.2) is *R* cos*γ* . The exterior normal at the boundary point *b* of the given interval has the same direction as the axe *t.* Then the angle between this normal and the axe *t* is zero, and its cosine is 1. But the exterior at the boundary point *a* of the given interval has the inverse direction*.* Therefore, the corresponding angle is π with value –1 of the cosine. Thus, the value at the right hand side of the equality (13.1) is the sum of the product of the considered function and the cosine of the angle between the exterior normal to the boundary and the coordinate axe. This has the same sense as the integral of the right hand side of the equality (13.2). Thus, the Gauss – Ostrogradsky formula is, in reality, the extension of the Newton – Leibniz formula to the multidimensional case.

Consider now three functions *P = P*(*x*,*y*,*z*), *Q = Q*(*x*,*y*,*z*), and *R = R*(*x*,*y*,*z*). We can write the equality (13.2) with replacing the function *R* and the variable *z* by *P* and *x* or *Q* and *y.* After summing these two equalities with (13.2), we get the following variant of the ***Gauss – Ostrogradsky formula***

  (13.3)

where*α* and *β*are the angles between the direction of the axes *x* and *y* respectively and the exterior normal to the surface *S.* This formula is the base of the analysis of the partial differential equations for the spatially multidimensional case.

### **13.2. Green formulas**

We have the formula (13.3). Determine here



where *u = u*(*x*,*y*,*z*), *v = v*(*x*,*y*,*z*) are smooth enough functions. Put it to the equality (13.3). We get

  (13.4)

Transform the values under the integral of the left hand-side. We have



The first term at the right hand-side of this equality includes the Laplace operator



The second term here is the scalar product of the vectors



that are the gradients of the considered functions. Therefore, we have



Note that the second multiplier under the integral of the right hand-side of the equality (13.4) is the normal derivative of the function *v*, i.e.



Thus, we transform the formula (13.4) to

  (13.5)

This is called the ***first Green formula***.

Changing the function *u* and *v*, we get



After summing two previous equalities, we obtain the ***second*** ***Green formula***

  (13.6)

### **13.3. Green function**

We determined in the previous lecture that the function *v =* 1/*r* is the solution of the Laplace equation (harmonic function), where *r* is the distance from the considered point and the origin. This is called the fundamental solution of the Laplace equation on the plan. Denote the origin by *M*0 = (*x*0,*y*0,*z*0) and the considered point by *M*= (*x*,*y*,*z*). We cannot to put this function *v* to the formula (13.6), because *r =* 0 if the point *M* is coincided with *M*0. Therefore, consider the ball *K*ε with center *M*0 and the small enough radium *ε*. Denote by Ωε the difference between the given set Ωε and the ball *K*ε. The function *v = v*(*M*) has the sense for all points of the set Ωε. Therefore, we can use the second Green formula with this value *v* after replacing the set Ω by Ωε. Note that Ωε has the exterior boundary *S* and interior boundary *S*ε that is the spherical surface with center *M*0 and the radium *ε*. Then we have the equality



The function *v* is harmonic in Ωε. Hence, we obtain

  (13.7)

The exterior normal at the sphere *S*ε has the inverse direction with respect to *r*. Then we have here



because *r = ε* on the sphere *S*ε. Find the integral



By the ***mean integral theorem***, the value of the surface integral is equal to the area of the surface and the value of the function under the integral at a point of this surface (mean value). The area of the spherical surface *S*ε is 4*πε*2. Then we transform the previous equality



where  is the mean value of the function *u* on the sphere *S*ε.

Using the equality  *r =ε* on *S*ε, we find the integral



by the mean integral theorem, where  is the mean value of the normal derivative of the function *u* on the sphere *S*ε.

Putting the results to the equality (13.7), we get

  (13.8)

Pass to the limit as *ε* tends to zero. Then the ball *K*ε tends to the point *M*0, and the set Ωε tends to Ω. Besides, the sphere *S*ε tends to the point *M*0 too. Therefore,  tends to *u*(*M*0). After passing to the limit in the equality (13.8), we get

  (13.9)

This is called the ***general integral Green formula***.

Consider the second Green formula with arbitrary harmonic function *v.* We get



After summing this equality with have

  (13.10)

where

  (13.11)

and  is, as before, the distance between the fixed point *M*0 = (*x*0,*y*0,*z*0) and the arbitrary point *M*= (*x*,*y*,*z*) that is the point of integration of the formula (13.10).

The function *G* is called the ***Green function***. It satisfies the Laplace equation

 Δ*G* = 0 in Ω (13.12)

because both functions in the right hand-side of the equality (13.11). We will use it for analysis of the boundary problem for the Laplace and the Poisson equations.

### **13.4. Green function method for the Dirichlet problem**

Consider the ***Laplace equation***

 Δ*u*(*M*)= 0, *M*∈Ω (13.13)

with first boundary condition

 *u*(*M*)= *ϕ*(*M*), *M*∈*S,* (13.14)

where the function *ϕ* is given. This problem is called the ***Dirichlet problem***.

Put the solution of this problem to the formula (13.10). We get

 . (13.15)

Suppose the function *G* satisfies the boundary condition

*G*(*M*,*M*0) = 0, *M*∈*S.*

By the equality (13.11), this formula is true, if the function *v* satisfies the following boundary condition

  (13.16)

Then from the formula (13.15), it follow that

  (13.17)

with integration by the variable *M.*

The formula (13.17) gives the direct dependence of the solution of the Dirichlet problem (13.13), (13.14) at the arbitrary point *M*0 from the known boundary function *ϕ.* Of course, it is necessary to know the Green function for using this formula. However, we can find it by the formula (13.11). The function *v* is harmonic, i.e. satisfies the Laplace equation

 Δ*v*(*M*)= 0, *M*∈Ω (13.18)

with boundary condition (13.16). Thus, if we solve the Dirichlet problem (13.18), (13.16) with concrete boundary function, we can find the solution of the boundary problem (13.13), (13.14) with arbitrary boundary function *ϕ.* This is the sense of the Green function method.

Consider now the ***Poisson equation***

 Δ*u*(*M*)= *f*(*M*), *M*∈Ω (13.19)

with boundary condition (13.14), where the function *f* is given. After putting its solution to the formula (13.10), we get



Choose here the previous function *G.* We obtain

  (13.20)

Now we found the solution of the Dirichlet problem for the Poisson equation at the arbitrary point. Thus, if we solved the Laplace equation (13.18) with concrete boundary condition (13.16), we can find the solution of the Poisson equation with arbitrary function *f* and arbitrary boundary function *ϕ*.

### **13.5. Green function method for the Neumann problem**

Consider now the Laplace equation

 Δ*u*(*M*)= 0, *M*∈Ω (13.20)

with second boundary condition. In this case, we know the derivative of the function *u* on the boundary. This is the normal derivative. Then we have the boundary condition

  (13.21)

where the function *ϕ* is given. This problem is called the ***Neumann problem***.

Put the solution of this problem to the formula (13.10). We get

 . (13.22)

Suppose the function *G* satisfies the boundary condition

 

By the equality (13.11), this formula is true, if the function *v* satisfies the following boundary condition

  (13.23)

Then from the formula (13.22), it follow that

 . (13.24)

The formula (13.24) gives the direct dependence of the solution of the Dirichlet problem (13.20), (13.21) at the arbitrary point *M*0 from the known boundary function *ϕ.* The Green function here is determined again by the formula (13.11), where the function *v* satisfies the Laplace equation

 Δ*v*(*M*)= 0, *M*∈Ω (13.25)

with boundary condition (13.23). Thus, if we solve the Neumann problem (13.20), (13.21) with concrete boundary function, we can find the solution of the boundary problem (13.20), (13.21) with arbitrary boundary function *ϕ.*

Consider now the ***Poisson equation***

 Δ*u*(*M*)= *f*(*M*), *M*∈Ω (13.26)

with boundary condition (13.21), where the function *f* is given. After putting its solution to the formula (13.10), we get



Choose here the previous function *G.* We obtain

  (13.27)

Now we found the solution of the Neumann problem for the Poisson equation at the arbitrary point. Thus, if we solved the Laplace equation (13.25) with concrete boundary condition (13.23), we can find the solution of the Poisson equation with arbitrary function *f* and arbitrary normal derivative function *ϕ*.

### **Conclusions**

* The Gauss – Ostrogradsky formula are extension of the formula of integration by parts to the multidimensional case.
* The Green formulas determine the relation between the spatial integral of two functions and the values of the Laplace operators here and boundary integral of the functions and the value of its normal derivatives.
* The Green function satisfies the Laplace equation with the concrete boundary condition.
* The fundamental solution of the Laplace equation on the space is used for the determination the Green function for three-dimensional case.
* By the Green function method, it is possible to find the solution of the boundary problem at the arbitrary point, if the Green function is known.
* The Green function of the Dirichlet problem for the Laplace equation does not depend from the given boundary function.
* The Green functions of the Dirichlet problems for the Laplace equation and the Poisson equation are same.
* All results are true for the Neumann problem too after changing the boundary condition for the Green function.
* All results are true for the two-dimensional case with changing the fundamental solution of the Laplace equation on the space by the fundamental solution of the Laplace equation on the plane for determining the Green function.